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SINGULAR SOLUTIONS OF THE STATIONARY LINEARIZED NAVIER-STOKES PROBLEM FOR MICROPOLAR VISCOUS LIQUIDS

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A system of singular solutions of the stationary linearized Navier-Stokes problem for viscous liquids characterized by an asymmetric stress tensor is constructed.

Micropolar (asymmetric) fluid mechanics concerns liquid media in which the stress tensor is nonsymmetric: $\sigma_{ij} \neq \sigma_{ji}$. Their study is stimulated, on the one hand, by a desire to refine the domains of applicability of classical hydrodynamics and, on the other hand, by the need to explain known experimental data and develop an adequate theory. The theoretical fundamental principles of micropolar fluid dynamics are rather fully discussed in [1], where resolving systems of differential equations are derived and typical problems solved within the framework of this theory are discussed. The equations of motion of micropolar liquids contain eight unknown functions, namely, three components of the linear velocity v, three components of the angular velocity $\mathbf{\Omega}$, pressure p, and density ρ, and because of their complexity they are integrated only in the simplest cases. Below, these equations are employed to describe the linear stationary Navier-Stokes problem when the number of unknown functions decreases to seven, and in this case a functional matrix is constructed whose columns represent a solution to the problem with a polar property.

These solutions may be used to construct a theory of hydrodynamic potentials for a description of stationary flows of micropolar viscous liquids using a scheme dating back to Odqvist [2, 3].

1. The equations of motion of a micropolar liquid in terms of the components of the linear and angular velocities are [1]

$$\rho \frac{d\mathrm{v}}{dt} = \rho \mathrm{f} - \mathrm{grad}\, p + (\lambda + 2\mu)\, \mathrm{grad\, div\, v} - (\mu - \gamma)\, \mathrm{rot\, rot\, v} - 2\gamma\, \mathrm{rot}\, \mathbf{\Omega} \quad ,$$

$$(\eta + \tau + \theta)\, \mathrm{grad\, div}\, \mathbf{\Omega} - \theta\, \mathrm{rot\, rot}\, \mathbf{\Omega} + 2\gamma \mathbf{\Omega} - \gamma\, \mathrm{rot\, v} + \rho \mathrm{m} = 0 , \qquad (1)$$

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\, (\rho \mathrm{v}) = 0 .$$

We shall consider the particular case of these equations corresponding to a linear steady-state flow:

$$(\lambda + 2\mu)\, \mathrm{grad\, div\, v} - (\mu - \gamma)\, \mathrm{rot\, rot\, v} - 2\gamma\, \mathrm{rot}\, \mathbf{\Omega} - \mathrm{grad}\, p + \rho \mathrm{f} = 0 ,$$

$$(\eta + \tau + \theta)\, \mathrm{grad\, div}\, \mathbf{\Omega} - \theta\, \mathrm{rot\, rot}\, \mathbf{\Omega} + 2\gamma \mathbf{\Omega} - \gamma\, \mathrm{rot\, v} + \rho \mathrm{m} = 0 ,$$

$$\mathrm{div\, v} = 0 .$$

After transformations, this system acquires the form

$$(\mu - \gamma)\, \Delta \mathrm{v} - 2\gamma\, \mathrm{rot}\, \mathbf{\Omega} - \mathrm{grad}\, p + \rho \mathrm{f} = 0 ,$$

$$\theta \Delta \mathbf{\Omega} + (\eta + \tau)\, \mathrm{grad\, div}\, \mathbf{\Omega} + 2\gamma \mathbf{\Omega} - \gamma\, \mathrm{rot\, v} + \rho \mathrm{m} = 0 , \qquad (2)$$

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$$- \operatorname{div} \mathrm{v} = 0 ,$$

or in matrix form

$$A\left(\frac{\partial}{\partial x}\right) U = F , \tag{3}$$

where

$$U = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \\ p \end{pmatrix}, \quad F = -\rho \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ 2m_1 \\ 2m_2 \\ 2m_3 \\ 0 \end{pmatrix}, \quad A\left(\frac{\partial}{\partial x}\right) = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} (\mu - \gamma)\,\Delta & 0 & 0 \\ 0 & (\mu - \gamma)\,\Delta & 0 \\ 0 & 0 & (\mu - \gamma)\,\Delta \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 2\gamma\partial_3 & -2\gamma\partial_2 & -\partial_1 \\ -2\gamma\partial_3 & 0 & 2\gamma\partial_1 & -\partial_2 \\ 2\gamma\partial_2 & -2\gamma\partial_1 & 0 & -\partial_3 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 2\gamma\partial_3 & -2\gamma\partial_2 \\ -2\gamma\partial_3 & 0 & 2\gamma\partial_1 \\ 2\gamma\partial_2 & -2\gamma\partial_1 & 0 \\ -\partial_1 & -\partial_2 & -\partial_3 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 2\theta\Delta + 2\kappa\partial_1^2 + 4\gamma & 2\kappa\partial_1\partial_2 & 2\kappa\partial_1\partial_3 & 0 \\ 2\kappa\partial_1\partial_2 & 2\theta\Delta + 2\kappa\partial_2^2 + 4\gamma & 2\kappa\partial_2\partial_3 & 0 \\ 2\kappa\partial_1\partial_3 & 2\kappa\partial_2\partial_3 & 2\theta\Delta + 2\kappa\partial_3^2 + 4\gamma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\partial_i = \frac{\partial}{\partial x_i}, \quad i = 1,\ 2,\ 3; \quad \kappa = \eta + \tau .$$

From energy considerations, the constraints for the coefficients of the system are as follows: $\mu > 0$, $3\lambda + 2\mu > 0$, $\mu - \gamma > 0$, $0 + \tau > 0$, $0 - \tau > 0$, $\gamma < 0$.

A distinguishing feature of system (3) is that it contains unknown functions with different orders of higher derivatives (second order for the linear and angular velocities and first order for the pressure). A symbolic determinant of system (3) is

$$\det A = -8\Delta^3\,[\theta(\mu - \gamma)\,\Delta + 2\gamma\mu]^2\,[(\kappa + \theta)\,\Delta + 2\gamma],$$

where

$$\Delta = \sum_{i=1}^{3} \partial_i^2 .$$

Therefore, despite the above distinctive feature, this system is of the elliptic type due to the indicated constraints on the domain of change of the coefficients entering it.

2. We now pass to construction of solutions of system (3) with a point singularity, i.e., solutions that fail to exist only at one point.

We denote by $\psi(x, y)$ the fundamental solution of the elliptical equation

$$-8\Delta^3 [\theta(\mu-\gamma)\Delta + 2\gamma\mu]^2 [(\kappa+\theta)\Delta + 2\gamma]\psi(x, y) = \delta(x-y). \tag{4}$$

Next, consider the following functional matrix:

$$\mathfrak{U}\psi = \| 4\Delta [\theta(\gamma-\mu)\Delta - 2\gamma\mu] a_{ij}(x, y) \|_{i,j=\overline{1,7}}, \tag{5}$$

where

$$\begin{gathered} a_{ij} = 2(\Delta\delta_{ij} - \partial_i\partial_j)(\theta\Delta + 2\gamma)[(\kappa+\theta)\Delta + 2\gamma]\psi(x, y), \\ a_{i+3,\, j+3} = \Big\{ [(\mu-\gamma)(\kappa+\theta)\Delta + 2\gamma(\mu-\gamma)]\delta_{ij}\Delta + \\ + \partial_i\partial_j[\kappa(\gamma-\mu)\Delta + 2\gamma^2]\Big\}\Delta\psi(x, y), \\ a_{i+3,j} = a_{i,j+3} = 2\gamma\sum_{k=1}^{3}\varepsilon_{ijk}\partial_k\Delta[(\kappa+\theta)\Delta + 2\gamma]\psi(x, y), \\ a_{7j} = a_{j7} = 2\partial_j\Delta[\theta(\gamma-\mu)\Delta - 2\gamma\mu][(\kappa+\theta)\Delta + 2\gamma]\psi(x, y), \\ a_{7,j+3} = a_{j+3,7} = 0, \\ a_{77} = -2(\mu-\gamma)\Delta^2[(\kappa+\theta)\Delta + 2\gamma][\theta(\mu-\gamma)\Delta + 2\gamma\mu]\psi(x, y), \\ i, j = \overline{1, 3}. \end{gathered} \tag{6}$$

Direct calculations show that the following equality holds:

$$A\left(\frac{\partial}{\partial x}\right)\mathfrak{U}\psi(x, y) = \delta(x-y)E. \tag{7}$$

Thus, it is necessary to represent the function $\psi(x, y)$ in explicit form in order to write out all elements of the matrix $\mathfrak{U}\psi(x, y)$. It follows from the form of the matrix $\mathfrak{U}\psi(x, y)$ that we may confine ourselves just to finding the function

$$\varphi(x, y) = 4\Delta[\theta(\gamma-\mu)\Delta - 2\gamma\mu]\psi(x, y), \tag{8}$$

which is a solution of the following equation:

$$-2\Delta^2[\theta(\gamma-\mu)\Delta - 2\gamma\mu][(\kappa+\theta)\Delta + 2\gamma]\varphi(x, y) = \delta(x-y) \tag{9}$$

or

$$2\theta(\mu-\gamma)(\kappa+\theta)\Delta^2(\Delta + k_1^2)(\Delta + k_2^2)\varphi(x, y) = \delta(x-y), \tag{10}$$

where

$$k_1^2 = \frac{2\gamma\mu}{\theta(\gamma-\mu)}, \quad k_2^2 = \frac{2\gamma}{\kappa+\theta}. \tag{11}$$

Hence we have

$$\Delta^2(\Delta + k_2^2)\varphi_1 = f_1, \quad \Delta^2(\Delta + k_1^2)\varphi_1 = f_2, \tag{12}$$

$$\Delta(\Delta + k_1^2)(\Delta + k_2^2)\varphi_1 = f_3, \quad (\Delta + k_1^2)(\Delta + k_2^2)\varphi_1 = f_4,$$

where

$$\varphi_1 = 2\theta(\mu-\gamma)(\kappa+\theta)\varphi, \quad f_i = \frac{-\exp(-\sigma_i r)}{4\pi r}, \quad i = 1, 2, 3; \quad f_4 = \frac{-1}{8\pi} r, \tag{13}$$

$$\sigma_1^2 = -k_1^2 = \frac{-2\gamma\mu}{\theta(\mu-\gamma)} > 0, \quad \sigma_2^2 = -k_2^2 = \frac{-2\gamma}{\kappa+\theta} > 0, \quad \sigma_3 = 0,$$

$$r = |x-y| = \sqrt{\left(\sum_{i=1}^{3}(x_i-y_i)^2\right)},$$

and therefore,

$$\varphi_1 = -\frac{f_1}{k_1^4(k_1^2-k_2^2)} + \frac{f_2}{k_2^4(k_1^2-k_2^2)} - \frac{f_3(k_1^2+k_2^2)}{k_1^4 k_2^4} + \frac{f_4}{k_1^2 k_2^2},$$

$$\Delta\varphi_1 = \frac{f_1}{k_1^2(k_1^2-k_2^2)} - \frac{f_2}{k_2^2(k_1^2-k_2^2)} + \frac{f_3}{k_1^2 k_2^2}, \tag{14}$$

$$\Delta^2\varphi_1 = -\frac{f_1}{k_1^2-k_2^2} + \frac{f_2}{k_1^2-k_2^2}, \quad \Delta^3\varphi_1 = \frac{k_1^2 f_1}{k_1^2-k_2^2} - \frac{k_2^2 f_2}{k_1^2-k_2^2}.$$

Now the sought matrix $\mathfrak{U}\psi$ may be rewritten in the form

$$2\theta(\mu-\gamma)(\kappa+\theta)\,\mathfrak{U}\psi(x, y) = \| a'_{ij}(x, y) \|_{i,j=\overline{1,7}}, \tag{15}$$

where

$$a'_{ij}(x, y) = 2(\Delta\delta_{ij} - \partial_i\partial_j)(\theta\Delta + 2\gamma)\,[(\kappa+\theta)\Delta + 2\gamma]\,\varphi_1(x, y),$$

$$a'_{i+3,\,j+3}(x, y) = \Big\{ (\mu-\gamma)\Delta\,[(\kappa+\theta)\Delta + 2\gamma]\,\delta_{ij} +$$

$$+ \partial_i\partial_j\,[\kappa(\gamma-\mu)\Delta + 2\gamma^2]\Big\}\,\Delta\varphi_1(x, y),$$

$$a'_{i+3,j}(x, y) = a'_{i,j+3}(x, y) = 2\gamma\sum_{k=1}^{3}\varepsilon_{ijk}\,\partial_k\,[(\kappa+\theta)\Delta + 2\gamma]\,\Delta\varphi_1(x, y),$$

$$a'_{7j}(x, y) = a'_{j7}(x, y) = 2\partial_j\,[\theta(\gamma-\mu)\Delta - 2\gamma\mu]\,[(\kappa+\theta) + 2\gamma]\,\Delta\varphi_1(x, y),$$

$$a'_{7,j+3}(x, y) = a'_{j+3,7}(x, y) \equiv 0, \tag{16}$$

$$a'_{77}(x, y) = -2(\mu-\gamma)[(\kappa+\theta)\Delta + 2\gamma][\theta(\mu-\gamma)\Delta + 2\gamma\mu]\Delta^2\varphi_1(x, y),$$

$$i, j = 1, 2, 3.$$

Introducing (8)-(13), (14) into (15), (16) and performing simple transformations we finally arrive at the following form of the sought matrix $\mathfrak{U}\psi(x, y)$:

$$\mathfrak{U}\psi(x, y) = \| C_{ij}(x, y) \|_{i,j=\overline{1,7}}, \tag{17}$$

where

$$C_{ij} = -\frac{1}{4\pi\mu}\left[\frac{\gamma}{\mu-\gamma}\,\frac{\exp(-\sigma_1 r)}{r} + \frac{1}{r}\right]\delta_{ij} - \frac{1}{8\pi\mu}\partial_i\partial_j\left[\frac{\theta}{\mu}\,\frac{\exp(-\sigma_1 r)-1}{r} - r\right],$$

$$C_{i+3,\,j+3} = -\frac{1}{8\pi\theta}\,\frac{\exp(-\sigma_1 r)}{r}\delta_{ij} - \frac{1}{16\pi}\partial_i\partial_j\left[\frac{\exp(-\sigma_1 r)-\exp(-\sigma_2 r)}{\gamma r} - \frac{\exp(\sigma_1 r)-1}{\mu r}\right],$$

$$C_{i+3,j} = C_{i,j+3} = -\frac{1}{8\pi\mu}\sum_{k=1}^{3}\varepsilon_{ijk}\partial_k\frac{\exp(-\sigma_1 r)-1}{r}, \tag{18}$$

$$C_{7,j+3} = C_{j+3,7} = 0, \quad C_{7j} = C_{j7} = \frac{1}{4\pi}\partial_j\frac{1}{r},$$

$$C_{77} = -(\mu-\gamma)\delta(x, y), \quad i, j = \overline{1, 3}; \quad \partial_i = \frac{\partial}{\partial x_i}.$$

Direct substitution of (17), (18) into (3) shows that the constructed matrix $\mathfrak{U}\psi$ consists of columns each of which is a solution of the initial system of differential equations (3) with $x \neq y$, $F \equiv 0$.

If there are no micropolar effects in the liquid, i.e., in the case of a symmetric stress tensor $\sigma_{ij} = \sigma_{ji}$ and an ordinary viscous fluid, $\gamma = 0$ and from (18) we have

$$v_{ij} = C^0_{ij} = -\frac{\delta_{ij}}{4\pi\mu r} + \frac{1}{8\pi\mu}\partial_i\partial_j r; \quad p_j = C^0_{7j} = -\frac{1}{4\pi}\partial_j\frac{1}{r}, \quad i, j = \overline{1, 3}.$$

This is the known Odqvist-Ladyzhenskaya solution of the linear stationary Navier-Stokes system corresponding to the action of a concentrated force directed along the x_j axis.

The constructed matrix $\mathfrak{U}\psi$ allows us to represent the particular solution of Eq. (3) in the form of the convolution

$$U(x) = (\mathfrak{U}\psi(x, y) * F(y))(x).$$

This formula may be written out using the well-known rule of multiplication of a matrix by a column.

To sum up, it should be noted that the method used for construction of the matrix $\mathfrak{U}\psi$ originates from [5-7], and its idea is most completely discussed in [7] as applied to various problems of the mechanics of a deformable solid.

NOTATION

σ_{ij}, components of the stress tensor; $\mathbf{v}(v_1, v_2, v_3)$, vector of the linear velocity; $\mathbf{\Omega}(\Omega_1, \Omega_2, \Omega_3)$, vector of the angular velocity; p, pressure; ρ, density; $\mathbf{f}(f_1, f_2, f_3)$, volume-distributed forces; $\mathbf{m}(m_1, m_2, m_3)$, volume-distributed moments; λ, μ, coefficients of volume and shear viscosity; η, τ, θ, coefficients of rotational viscosity; γ, measure of "coalescence" of a liquid particle with its environment; δ_{ij}, Kronecker symbol; $\delta_{ii} = 1$; $\delta_{ij} = 0$, $i \neq j$; ε_{ijk}, Levi-Civit a symbol; $\varepsilon_{ijk} = +1$ or $\varepsilon_{ijk} = -1$ if i, j, k form an even or odd permutation of the numbers 1, 2, 3, $\varepsilon_{ijk} = 0$ if $i = j$; $i = k$; $j = k$; $\delta(x - y)$, Dirack delta-function; E, unit matrix; $x(x_1, x_2, x_3)$, point of three-dimensional Euclidean space $\partial_i = \partial/\partial x_i$; $\partial_i\partial_j = \partial^2/\partial x_i \partial x_j$; $\Delta = \sum_{i=1}^{3} \partial^2/\partial x_i^2$, Laplacian; $r = \sqrt{(\sum_{i=1}^{3} (x_i - y_i)^2)}$.

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